# ECO 391 Economics and Business Statistics

*Lecture 2: Continuous Probability Distributions and The Normal Distribution*

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# Overview

1. Continuous Random Variables

2. The Normal Distribution

## Continuous Random Variables

# Continuous Random Variables

- Remember that random variables may be classified as
  - Discrete: The random variable assumes a countable number of distinct values.
  - Continuous: The random variable is characterized by (infinitely) uncountable values within any interval.
- When computing probabilities for a continuous random variable, keep in mind that $P(X = x) = 0$. Why?
  - We cannot assign a nonzero probability to each infinitely uncountable value and still have the probabilities sum to one.
  - Thus, since $P(X = a)$ and $P(X = b)$ both equal zero, the following holds for continuous random variables:

$$P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b).$$

# PDF and CDF

- Probability Density Function (PDF) $f(X)$ of a continuous random variable $X$ describes the relative likelihood that $X$ assumes a value within a given interval (e.g., $P(a \leq X \leq b)$), where
  - $f(x) > 0$ for all possible values of $X$.
  - The **area** under $f(x)$ over all values of $x$ equals one.
- Cumulative Density Function (CDF) $F(x)$ of a continuous random variable $X$
  - For any value $x$ of the random variable $X$, the cumulative distribution function $F(x)$ is computed as
$$F(x) = P(X \leq x).$$
  - As a result,
$$P(a \leq X \leq b) = F(b) - F(a).$$

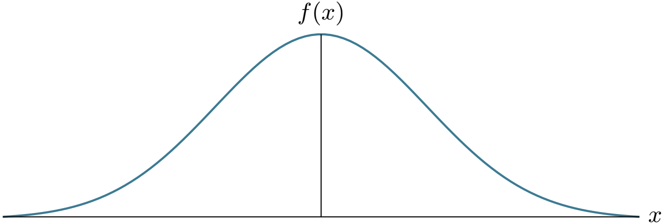
# The Normal Distribution

# The Normal Distribution

- Symmetric
- Bell-shaped
- Closely approximates the probability distribution of a wide range of random variables, such as the
  - Heights and weights of newborn babies
  - Scores on SAT
  - Cumulative debt of college graduates
- Serves as the cornerstone of statistical inference.

# The Normal Distribution

- **Symmetric** about its mean: Mean=Median=Mode




- **Asymptotic**: that is, the tails get closer and closer to the horizontal axis, but never touch it.

# The Normal Distribution

- The normal distribution is completely described by two parameters: $\mu$ and $\sigma^2$.
  - $\mu$ is the population mean which describes the central location of the distribution.
  - $\sigma^2$ is the population variance which describes the dispersion of the distribution.
- Probability density function of the normal distribution for a random variable $X$ with mean $\mu$ and variance $\sigma^2$:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

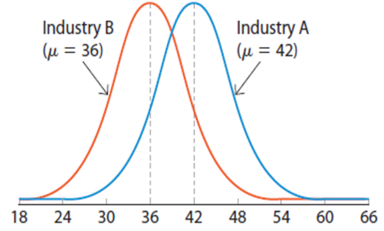
where $\pi = 3.14$, $\exp(x) = e^x$, and $e \approx 2.718$ is the base of the natural logarithm.

# The Normal Distribution

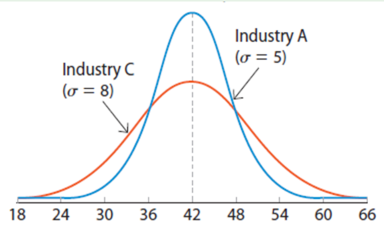
## Example

Suppose the ages of employees in Industries $A$, $B$, and $C$ are normally distributed. Here are the relevant parameters:

| | Industry A | Industry B | Industry C |
|---|---|---|---|
| $\mu$ | 42 | 36 | 42 |
| $\sigma$ | 5 | 5 | 8 |



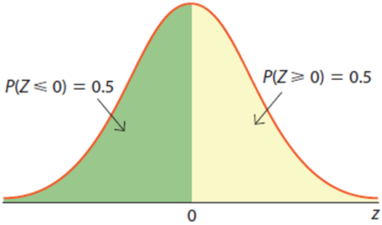

(a) $\sigma$ is the same, $\mu$ is different




(b) $\mu$ is the same, $\sigma$ is different

# The Standard Normal $Z$ Distribution

- A special case of the normal distribution is the standard normal $Z$ distribution:
  - Mean $\mu = 0$.
  - Standard deviation $\sigma = 1$.



# Standard Normal Table ($z$ Table)

- Gives the cumulative probabilities $P(Z < z)$ for positive and negative values of $z$.
- Since the random variable $Z$ is symmetric about its mean of 0,

$$P(Z < 0) = P(Z > 0) = 0.5$$

- To obtain the $P(Z < z)$, read down the $z$ column first, then across the top.

# Standard Normal Table ($Z$ Table)

Table for positive *z* values.

| *z* | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 |
|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 |

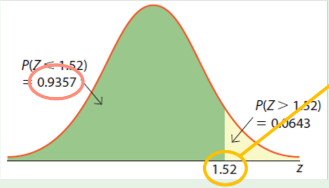
Table for negative *z* values.

| *z* | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 |
|---|---|---|---|---|---|
| -3.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| -3.8 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| -3.7 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| -3.6 | 0.0002 | 0.0002 | 0.0001 | 0.0001 | 0.0001 |
| -3.5 | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |

## Example

Finding the Probability for a Given $z$ Value.

- Transform normally distributed random variables into standard normal random variables and use the $z$ table to compute the relevant probabilities.
- The $z$ table provides cumulative probabilities $P(Z \leq z)$ for a given $z$.



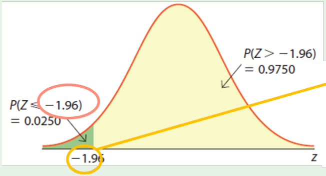

Portion of right-hand page of $z$ table.
If $z$ = 1.52, then look up

| $z$ | 0.00 | 0.01 | 0.02 |
|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | ↓ |
| 0.1 | 0.5398 | 0.5438 | ↓ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 1.5 | → | → | 0.9357 |

## Example

Finding the Probability for a Given $z$ Value.

- Remember that the $z$ table provides cumulative probabilities $P(Z < z)$ for a given $z$.
- If $z$ is negative, we can look up this probability from the left-hand page of the $z$ table



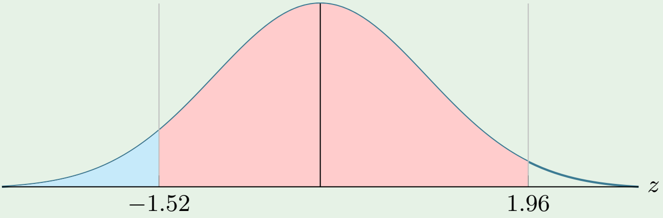

Portion of left-hand page of $Z$ Table.
If $z$ = -1.96, then look up

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 |
|---|---|---|---|---|---|---|---|
| −3.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | ↓ |
| −3.8 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | ↓ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| −1.9 | → | → | → | → | → | → | 0.0250 |

# More Example

## Example

Finding probabilities for a standard normal random variable $Z$ that $P(-1.52 \leq Z \leq 1.96)$.




$$
\begin{aligned}
&P(-1.52 \leq Z \leq 1.96) \\
=&P(Z \leq 1.96) - P(Z \leq -1.52) \\
=&0.9750 - 0.0643 \\
=&0.9107
\end{aligned}
$$

# Finding a $z$ Value for a Given Probability

- For a standard normal variable $Z$, find the $z$ values that satisfy $P(Z \leq z) = 0.6808$.

| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| –3.4 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 |
| –3.3 | .0005 | .0005 | .0005 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 |
| –3.2 | .0007 | .0007 | .0006 | .0006 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 |
| –3.1 | .0010 | .0009 | .0009 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 |
| –3.0 | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| –2.9 | .0019 | .0018 | .0018 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| –2.8 | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| –2.7 | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| –2.6 | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| –2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| –2.4 | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| –2.3 | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| –2.2 | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| –2.1 | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| –2.0 | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| –1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| –1.8 | .0359 | .0351 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| –1.7 | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| –1.6 | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| –1.5 | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| –1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |
| –1.3 | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| –1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| –1.1 | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| –1.0 | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| –0.9 | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| –0.8 | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| –0.7 | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| –0.6 | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| –0.5 | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| –0.4 | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| –0.3 | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3483 |
| –0.2 | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| –0.1 | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| –0.0 | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |

# Finding a $z$ Value for a Given Probability

- For a standard normal variable $Z$, find the $z$ values that satisfy $P(Z \leq z) = 0.6808$.

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

# Finding a $z$ Value for a Given Probability

- Go to the standard normal table and find 0.6808 in the body of the table.
- Find the corresponding $z$ value from the row/column of $z$.
- $z = 0.47$.

# The Normal Transformation

Any normally distributed random variable $X$ with mean $\mu$ and standard deviation $\sigma$ can be transformed into the standard normal random variable $Z$ as

$$Z = \frac{X - \mu}{\sigma}$$

with corresponding values

$$z = \frac{x - \mu}{\sigma}.$$

As constructed, the mean of $Z$ is 0 and standard deviation of $Z$ is 1.

# The Normal Transformation

- A $z$ value specifies by how many standard deviations the corresponding $x$ value falls above ($z > 0$) or below ($z < 0$) the mean.
- A positive $z$ indicates by how many standard deviations the corresponding $x$ lies above $\mu$.
- A zero $z$ indicates that the corresponding $x$ equals $\mu$.
- A negative $z$ indicates by how many standard deviations the corresponding $x$ lies below $\mu$.

# The Normal Transformation

- Use the **inverse transformation** to compute probabilities for given $x$ values.
- A standard normal variable $Z$ can be transformed to the normally distributed random variable $X$ with mean $\mu$ and standard deviation $\sigma$ as

$$X = \mu + Z\sigma$$

with corresponding values $x = \mu + z\sigma$.

# The Normal Transformation

## Example

Scores on a management aptitude exam are normally distributed with a mean of 72 ($\mu$) and a standard deviation of 8 ($\sigma$).

- Given this information, we can transform any score into a $z$ value which shows its relative location to the mean.
- For instance, given the score 60, using the transformation formula,

$$z = \frac{x - \mu}{\sigma} = \frac{60 - 72}{8} = -1.5$$

Using the standard normal table, find

$$P(Z > -1.5) = 1 - P(Z < -1.5) = 1 - 0.0668 = 0.9332$$

# Summary

- PDF and CDF.
- The normal distribution: mean and variance.
- The standard normal distribution and transformation from the normal.